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Hydromagnetic spin-up of a fluid confined by two flat electrically conducting boundaries

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The prototype linear spin-up problem consisting of a homogeneous viscous electrically conducting fluid confined between two infinite flat rotating electrically conducting plates in the presence of an applied axial magnetic field is studied in an effort to understand better the strength and nature of the coupling between a fluid and its rotating conducting container. It is assumed that the response time of the bounding plates to a magnetic perturbation is much less than the fluid spin-up time and that the plate conductivity is an arbitrary function of distance from the fluid–plate interface. The general Laplace transform solution is inverted and discussed for three special cases: magnetic diffusion regions thick compared with fluid depth during spin-up, arbitrary magnetic field strength and boundary conductance; magnetic diffusion regions thin, weak conductance, arbitrary field; magnetic diffusion regions thin, strong conductance, arbitrary field. In each case conductance of the boundary strengthens the coupling between fluid and boundary, thereby decreasing the spin-up time. The corresponding single plate analysis of Loper (1970*a*) is found to predict spin-up accurately only if the boundary conductance is much smaller than that of the fluid. The fluid possesses an oscillatory mode of spin-up if the magnetic diffusion regions are thin and boundary conductance is large. That is, the inviscid current-free core of fluid rotates significantly faster than the boundaries during a portion of the spin-up process.

1. Introduction

The spin-up of a fluid contained between two infinite flat rotating plates is a prototype for a large class of motions of contained fluids. The linearized spin-up of an incompressible viscous fluid was successfully analysed by Greenspan & Howard (1963), yielding the spin-up time for the fluid and elucidating the role of the Ekman layers and secondary flow in the spin-up process.

Loper & Benton (1970) generalized the work of Greenspan & Howard (1963) by considering the linear spin-up of an electrically conducting fluid bounded by two infinite flat rotating electrically insulating plates in the presence of an applied magnetic field. They found that hydromagnetic effects strengthen the coupling between fluid and boundary; the spin-up time decreases with increasing magnetic field strength. In addition, they clarified the role of the Ekman–Hartmann boundary layers and the magnetic diffusion regions in the spin-up process.

The present analysis is motivated by a desire to understand better the strength and nature of the coupling between a rotating electrically conducting fluid and its electrically conducting boundaries. To this end, a prototype linear hydromagnetic spin-up problem is investigated wherein a homogeneous electrically conducting fluid is bounded axially by two electrically conducting plates in the presence of an applied axial magnetic field. Whereas in the insulating plate analysis of Loper & Benton (1970) electric currents are confined to the Ekman–Hartmann layers and magnetic diffusion regions, in the present analysis, currents may flow within the plates also. It is seen below that such currents have the effect of increasing the strength of coupling between fluid and boundary. (For a detailed discussion of the character of and interactions between the various regions of the flow, the reader is referred to Loper (1970*b*).)

Greenspan & Howard (1963) demonstrated the success of the boundary-layer approach in analysing the spin-up problem; i.e. that Ekman suction found from a single plate analysis can be applied to an inviscid layer of fluid to yield the spin-up time for the two plate problem. Loper & Benton (1970) found the boundary-layer approach to be valid in the insulating plate problem, regardless of the thickness of the magnetic diffusion regions. In his analysis of the steady boundary layer on a single conducting flat boundary, Loper (1970*a*) obtained a spin-up time for the corresponding two plate problem by assuming the boundary-layer approach to be valid for conducting boundaries. It is seen below that this approach is successful for the conducting plate problem *only* if the plate conductance is much less than that of the fluid.

In §2, the problem is linearized with respect to Rossby number, $\Delta\Omega/\Omega = \epsilon$, and simplified by assuming the response time of the boundaries to a magnetic perturbation is much less than the spin-up time of the fluid. The resulting four parameter problem is solved in the Laplace transform plane. It is found that the finite conductance of the contained fluid removes the singularity (as plate conductance becomes large) found by Loper (1970*a*) in the corresponding single plate analysis. The general solution obtained in §2 is simplified and inverted in §§3 and 4 by assuming the magnetic diffusion regions to be thick and thin, respectively, compared with the fluid depth during spin-up. In each case, conductance of the boundaries acts to strengthen the coupling between fluid and boundaries, thereby decreasing the spin-up time. This strengthening may be explained by the fact that electric current loops circulating between the conducting boundaries and magnetic diffusion regions act in the same sense as those circulating between the Ekman–Hartmann layers and magnetic diffusion regions. Also, in each case, the finite conductance of the fluid becomes important and significant deviations from the behaviour predicted by Loper (1970*a*) occur if $C_p \geqslant O(C_f)$, where C_p and C_f are conductance of boundary and fluid respectively. In §4, a new oscillatory mode of spin-up is found to occur when the plate conductance is sufficiently strong.

2. Formulation and general solution

Consider an incompressible viscous fluid of constant electrical conductivity σ_f filling the gap of width $2d$ between two infinite flat plates of arbitrary electrical conductivity $\sigma_p\,(z)$. With the axial co-ordinate z measured from the mid depth, $\sigma_p\,(z)$ is defined only for $d \leqslant |z| < \infty$. To preserve the symmetry of the problem about the plane $z = 0$, assume σ_p is an even function of z. At time zero, the fluid is in rigid body rotation with angular speed Ω, the plates co-rotate in the same direction with speed $\Omega(1+\epsilon)$, where $|\epsilon| \ll 1$, and a uniform magnetic field of strength B_0 is imposed everywhere normal to planes z = constant (see figure 1). This initial-value problem is a direct extension of the non-magnetic analysis of Greenspan & Howard (1963) and of the insulating plate analysis of Loper & Benton (1970) and includes those analyses as special cases. As in those papers, attention is focused upon the time needed for the fluid to spin up to the speed of the plates as well as upon the manner in which the spin-up is accomplished.

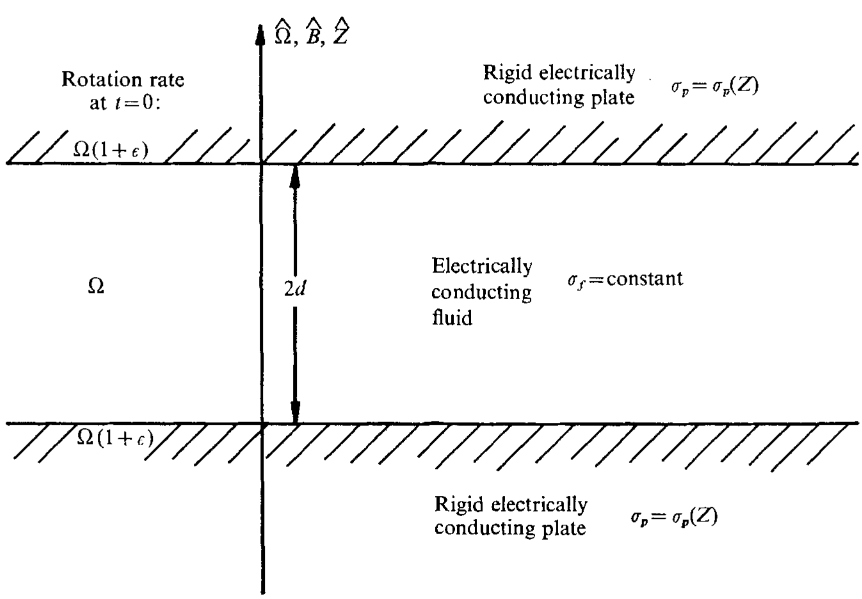

FIGURE 1. Schematic depiction of configuration and initial conditions.

The equations governing the velocity $\mathbf{v}$, pressure π and magnetic field $\mathbf{B}$ within the fluid may be non-dimensionalized and combined by use of complex notation following Loper & Benton (1970) to yield (27) and (28) of that analysis. Similarly, the magnetic field within the plates may be expressed as

$$\mathbf{B}_p(r, z, t) = B_0\hat{z} + B_0\mu\sigma_f(\nu\Omega)^{\frac{1}{2}}\epsilon[rA_p(\xi, \tau)\hat{r} + rB_p(\xi, \tau)\hat{\theta} + dC_p(\xi, \tau)\hat{z}], \quad (1)$$

where

$$\tau = \Omega t, \quad \xi = z/L, \quad L = \sigma_p^{-1}(d)\int_d^\infty \sigma_p(z)\,dz,$$

r, t, μ and ν are, respectively, dimensionless time, dimensionless axial co-ordinate, length scale within the plates, radial co-ordinate, time, magnetic permeability, and kinematic viscosity. Introduction of complex notation

$$\left.\begin{aligned} F(\zeta, \tau) &= U(\zeta, \tau) + iV(\zeta, \tau), \\ M(\zeta, \tau) &= A(\zeta, \tau) + iB(\zeta, \tau), \\ N(\xi, \tau) &= A_p(\xi, \tau) + iB_p(\xi, \tau), \end{aligned}\right\} \tag{2}$$

where U, V, A, B and ζ are, respectively, dimensionless radial velocity, azimuthal velocity, radial field, azimuthal field and z/d allows the linearized problem to be expressed as

$$F_\tau - EF_{\zeta\zeta} + 2iF = 2\alpha^2 E^{\frac{1}{2}} M_\zeta - P(\tau), \tag{3}$$

$$\delta M_\tau - EM_{\zeta\zeta} = E^{\frac{1}{2}} F_\zeta, \tag{4}$$

$$\sigma^2 R N_\tau - \sigma N_{\xi\xi} + \sigma_\xi N_\xi = 0, \tag{5}$$

with conditions

$$F(\zeta, 0) = M(\zeta, 0) = N(\xi, 0) = 0, \qquad F(\pm 1, \tau) = i,$$

$$M(\pm 1, \tau) = N(\pm d/L, \tau), \qquad E^{\frac{1}{2}} \phi M_\zeta(\pm 1, \tau) = N_\xi(\pm d/L, \tau),$$

$$N(\pm\infty, \tau) = 0, \qquad \mathrm{Re}\left[\int_0^1 F(\zeta, \tau)\, d\zeta\right] = 0,$$

where

$$\alpha = B_0(\sigma_f/2\rho\Omega)^{\frac{1}{2}} = \text{magnetic interaction parameter},$$

$$\delta = \sigma_f \mu\nu = \text{magnetic Prandtl number},$$

$$E = \nu/\Omega d^2 = \text{Ekman number},$$

$$\phi = (\Omega/\nu)^{\frac{1}{2}} \sigma_f^{-1} \int_d^\infty \sigma_p(z)\, dz = \text{conductance ratio},$$

$$R = \mu\Omega\sigma_p^{-1}(d)\left[\int_d^\infty \sigma_p(z)\, dz\right]^2 = \text{magnetic Reynolds number},$$

and

$$\sigma(\xi) = \sigma_p^{-1}(d)\, \sigma_p(z).$$

The magnetic Reynolds number R is a measure of the response time of the magnetic field within the plates. In order not to obscure the spin-up dynamics unnecessarily, it is assumed that this response time is much shorter than the spin-up time, i.e.

$$R \ll \tau_s, \tag{6}$$

where τ_s is the spin-up time. This approximation is very good for a typical laboratory experiment but is marginal (i.e. $R \approx \tau_s$) for the interior of the earth. The influence of the response time of the magnetic field upon the spin-up time has been investigated by Loper (1970*b*).

With assumption (6) the five parameter problem defined above reduces to a four parameter problem. What is more, one dependent variable $[N(\xi, \tau)]$ may be eliminated from direct consideration, yielding a much simpler set of equations. If the unsteady term in (5) is neglected, that equation may be directly integrated for an arbitrary function $\sigma(\epsilon)$ to yield

$$N(\xi, \tau) = M_0(\tau) \operatorname{sgn}(\xi) \left[1 - \int_{d/L}^{|\xi|} \sigma(\xi)\, d\xi\right], \tag{7}$$

where $M_0(\tau)$ is the value of M (and N) at $\xi = +d/L$, $\zeta = +1$. The problem now reduces to (3) and (4) subject to conditions

$$F(\zeta, 0) = M(\zeta, 0) = 0, \quad F(\pm 1, \tau) = i,$$

$$\operatorname{Re}\left[\int_0^1 F(\zeta, \tau)\, d\zeta\right] = 0, \quad M(\pm 1, \tau) = \mp E^{\frac{1}{2}} \phi M_\zeta(\pm 1, \tau).$$

In the limit $\alpha \to 0$, the prototype spin-up problem of Greenspan & Howard (1963) is recovered while the limit $\phi \to 0$ yields the insulating plate problem of Loper & Benton (1970).

The Laplace transform of (3) and (4) together with the above conditions yields

$$E\bar{F}_{\zeta\zeta} - (2i + s)\bar{F} = -2\alpha^2 E^{\frac{1}{2}} \bar{M}_\zeta + \bar{P}(s), \tag{8}$$

$$E\bar{M}_{\zeta\zeta} - \delta S \bar{M} = -E^{\frac{1}{2}} \bar{F}_\zeta, \tag{9}$$

subject to

$$\bar{F}(\pm 1, s) = i/s, \quad \bar{M}(\pm 1, s) = \mp E^{\frac{1}{2}} \phi \bar{M}_\zeta(\pm 1, s), \quad \operatorname{Re}\left[\int_0^1 \bar{F}(\zeta, s)\, d\zeta\right] = 0.$$

The solutions for $\bar{F}$, $\bar{M}$, $\bar{N}$ and $\bar{P}$ which satisfy (7), (8) (9) and the appropriate boundary and initial conditions are

$$\bar{F}(\zeta, s) = \frac{i(s-2i)}{sD}[\tilde{\eta}\theta - \eta\tilde{\theta}] + \frac{2i}{D}[E^{-\frac{1}{2}}\tilde{\theta} - \tilde{\eta}] \Big[(2i + s - m^2)(1 + \phi m \coth[mE^{-\frac{1}{2}}]) \times k \frac{\cosh(kE^{-\frac{1}{2}}\zeta)}{\sinh(kE^{-\frac{1}{2}})} - (2i + s - k^2)(1 + \phi k \coth[kE^{-\frac{1}{2}}])\, m \frac{\cosh(mE^{-\frac{1}{2}}\zeta)}{\sinh(mE^{-\frac{1}{2}})}\Big], \tag{10}$$

$$\bar{M}(\zeta, s) = -\frac{2i}{D}(s + 2i)[E^{-\frac{1}{2}}\tilde{\theta} - \tilde{\eta}] \Big[(1 + \phi m \coth[mE^{-\frac{1}{2}}]) \frac{\sinh(kE^{-\frac{1}{2}}\zeta)}{\sinh(kE^{-\frac{1}{2}})} - (1 + \phi k \coth[kE^{-\frac{1}{2}}]) \frac{\sinh(mE^{-\frac{1}{2}}\zeta)}{\sinh(mE^{-\frac{1}{2}})}\Big], \tag{11}$$

$$\bar{N}(\xi, s) = \operatorname{sgn}(\xi) \frac{2i\phi}{D}(s + 2i)[E^{-\frac{1}{2}}\tilde{\theta} - \tilde{\eta}][k \coth(kE^{-\frac{1}{2}}) - m \coth(mE^{-\frac{1}{2}})] \times \left[1 - \int_{d/L}^{|\xi|} \sigma(\xi)\, d\xi\right], \tag{12}$$

$$\bar{P}(s) = -i \frac{(s^2 + 4)}{sD}[\tilde{\eta}\theta - \eta\tilde{\theta}], \tag{13}$$

where $$D = 2sE^{-\frac{1}{2}}\theta\tilde{\theta} - (s+2i)\theta\tilde{\eta} - (s-2i)\tilde{\theta}\eta, \tag{14}$$

$$\theta = k(s+2i-m^2)\coth(kE^{-\frac{1}{2}}) - m(s+2i-k^2)\coth(mE^{-\frac{1}{2}}) + \phi km(k^2-m^2)\coth(kE^{-\frac{1}{2}})\coth(mE^{-\frac{1}{2}}), \tag{15}$$

$$\eta = k^2 - m^2 + \phi m(s+2i-m^2)\coth(mE^{-\frac{1}{2}}) - \phi k(s+2i-k^2)\coth(kE^{-\frac{1}{2}}), \tag{16}$$

$$k = [n + (n^2-q^2)^{\frac{1}{2}}]^{\frac{1}{2}}, \quad \mathrm{Re}(k) \geqslant 0, \tag{17}$$

$$m = [n - (n^2-q^2)^{\frac{1}{2}}]^{\frac{1}{2}}, \quad \mathrm{Re}(m) \geqslant 0, \tag{18}$$

$$n = \tfrac{1}{2}(s+2i+2\alpha^2+\delta s), \tag{19}$$

$$q^2 = \delta s(s+2i) \tag{20}$$

and a tilde indicates a complex conjugate with s regarded as real.

Inspection of these solutions reveals that they are meromorphic; that is, branch cuts do not contribute to the inversion. Contributions to the inversions of (10)–(13) come from a pole at $s = 0$, representing the final steady state, and from zeros of the function D. It is anticipated that, at least for a limited range of the parameters, there is an isolated zero of D on the negative real axis representing the spin-up mode. (For other values of the parameters, a pair of conjugate spin-up poles may exist.) The inversion of the residues of these two poles (i.e. steady pole at $s = 0$ and spin-up pole) is of primary interest. In addition, two pairs of conjugate sequences of zeros of D, representing inertial oscillations, Alfvén waves and boundary-layer formation, are anticipated. These poles will be ignored when possible (as in §3) or approximated by a branch cut when necessary (as in §4).

In the case of a single conducting plate (Loper 1970*a*) the limit $\phi \to \infty$ is singular, with the result that the analysis predicts magnetic perturbations of infinite amplitude and instantaneous spin-up of the corresponding confined fluid. Inspection of (10)–(20) reveals that no such singularity occurs in the corresponding analysis of the case of two conducting plates; magnetic perturbations remain finite as $\phi \to \infty$ and, as will be seen below, spin-up is not instantaneous. This contradiction may be clarified as follows. In the limit $\phi \to \infty$, the plate acts as a perfect conductor and the imposed magnetic field is 'frozen in' (see Cowling 1957). At the same time, for the single plate configuration the semi-infinite extent of fluid also acts as a perfect conductor and forces the magnetic field to rotate with it. Since the semi-infinite fluid never spins up, the differential rotation steadily twists the imposed field into the azimuthal direction resulting in an azimuthal field which grows without bound. The singularity does not appear in the two plate problem for two reasons. First, conductance of the contained fluid ($2d\sigma_f$) is finite (the field lines can slip through the fluid) and second, the fluid itself spins up so that the differential rotation which causes the twisting does not last indefinitely. It is seen below that an important measure of plate conductance for the spin-up problem is

$$\phi E^{\frac{1}{2}} = d^{-1}\sigma_f^{-1}\int_d^\infty \sigma_p(z)\,dz = C_p/C_f.$$

In physical terms, $\phi E^{\frac{1}{2}}$ is the ratio of the conductance of the plates to that of the entire fluid (ϕ itself measures plate conductance with respect to the conductance of a layer of fluid one Ekman depth in thickness). When $\phi E^{\frac{1}{2}} \geqslant O(1)$, the finite conductance of the fluid becomes important and significant deviations from the behaviour predicted by the single plate analysis of Loper (1970*a*) occur.

The original five parameter problem has been reduced to a four parameter problem by virtue of assumption (6). Ranges of interest for the remaining parameters are $E \ll 1$, $\delta \ll 1$, and α and ϕ arbitrary. In order to reduce the four parameter problem effectively to a two parameter problem wherein α and ϕ are ordered with respect to E, attention will be restricted to the two extremes of magnetic diffusion regions thick during the spin-up phase (in §3) and thin during the spin-up phase (in §4).

3. Magnetic diffusion regions thick during spin-up

If α is not too large, the magnetic diffusion regions grow at the resistive diffusion rate, given in dimensionless terms as $\zeta = \tau^{\frac{1}{2}}E^{\frac{1}{2}}\delta^{-\frac{1}{2}}$. If the magnetic Prandtl number δ is sufficiently small that

$$\delta \ll \tau_s E, \tag{21}$$

where τ_s is the spin-up time, the magnetic diffusion regions diffuse across the fluid much more quickly than the fluid spins up and these regions are thick during the spin-up phase. This allows the time development of the magnetic diffusion regions to be ignored. In mathematical terms, all poles produced by zeros of D associated with coth $(mE^{-\frac{1}{2}})$ and $\coth(\tilde{m}E^{-\frac{1}{2}})$ may be neglected. Provided that $E \ll 1$ and that the spin-up pole is not located near $s = \pm 2i - 2\alpha^2$, the following simplified expressions for the fluid velocity variable $\bar{F}(\zeta, s)$ and the perturbation magnetic field at the fluid–plate interface are obtained.

$$\bar{F}(\zeta, s) = \frac{i}{s} + \frac{2ik_s\tilde{k}_s}{E^{\frac{1}{2}}\Delta}\left[\frac{\cosh(k_sE^{-\frac{1}{2}}\zeta)}{\sinh(k_sE^{-\frac{1}{2}})} - 1\right], \tag{22}$$

$$\bar{M}(1, s) = \bar{N}(d/L, s) = \frac{2i\phi}{1+\phi E^{\frac{1}{2}}}\frac{k_s\tilde{k}_s}{E^{\frac{1}{2}}\Delta}, \tag{23}$$

where

$$\Delta = 2E^{-\frac{1}{2}}k_s\tilde{k}_s\left(s + 2\alpha^2\frac{\phi E^{\frac{1}{2}}}{1+\phi E^{\frac{1}{2}}}\right) - 2i(k_s - \tilde{k}_s) + \left(2\alpha^2\frac{1-\phi E^{\frac{1}{2}}}{1+\phi E^{\frac{1}{2}}} - s\right)(k_s + \tilde{k}_s) - \frac{4\alpha^2 E^{\frac{1}{2}}}{1+\phi E^{\frac{1}{2}}}, \tag{24}$$

and

$$k_s = (2i + 2\alpha^2 + s)^{\frac{1}{2}}. \tag{25}$$

Note that this simplified solution for $\bar{F}$ still satisfies all the prescribed initial and boundary conditions and also yields the anticipated final value, $F(\zeta, \infty) = i$.

The inversion of $\bar{F}$ from (22) has contributions from the steady pole and the spin-up pole, which are the primary concern here, and from branch cuts representing boundary-layer formation and inertial oscillations, which will be neglected. With $E \ll 1$, it may be seen from (24) that a zero of Δ occurs close to $s = -2\alpha^2\phi E^{\frac{1}{2}}(1+\phi E^{\frac{1}{2}})^{-1}$. It is easily verified that the zero which yields the spin-up pole is in fact located at

$$s_p = -2\alpha^2 \frac{\phi E^{\frac{1}{2}}}{1+\phi E^{\frac{1}{2}}} - E^{\frac{1}{2}}\lambda \tag{26}$$

correct to order $E^{\frac{1}{2}}$, where

$$\lambda = \left[\frac{\alpha^2}{1+\phi E^{\frac{1}{2}}} + \left(1+\frac{\alpha^4}{(1+\phi E^{\frac{1}{2}})^2}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}. \tag{27}$$

The function λ is a generalization of the function $\beta = [\alpha^2+(1+\alpha^4)^{\frac{1}{2}}]^{\frac{1}{2}}$ which appears in the insulated plate analysis. The location of the spin-up pole may be checked in several limits:

(*a*) If $\alpha = 0$, $s_p = -E^{\frac{1}{2}}$ in agreement with Greenspan & Howard (1963).

(*b*) If $\phi = 0$, $s_p = -\beta E^{\frac{1}{2}}$ in agreement with Loper & Benton (1970).

(*c*) If $\phi E^{\frac{1}{2}} \ll 1$, $s_p = -E^{\frac{1}{2}}(\beta+2\alpha^2\phi)$ in agreement with Loper (1970*a*).

In the limit of highly conducting plates ($\phi E^{\frac{1}{2}} \gg 1+\alpha^2$) the location of the spin-up pole becomes independent of the plate conductivity:

$$\lim_{\phi\to\infty} s_p = -2\alpha^2 - E^{\frac{1}{2}}.$$

In this limit, the finite resistivity of the fluid (due to the finite depth) acts to limit the flow of electrical current, thus giving a finite value, in contrast to the approximate calculation of Loper (1970*a*) (valid for $\phi E^{\frac{1}{2}} \ll 1$) which predicts no finite limiting value.

If branch cuts which give boundary-layer formation and inertial oscillations are neglected, the inversion of (22) is

$$F(\zeta, \tau) = i - i\{1-\exp[-E^{-\frac{1}{2}}(\lambda+i/\lambda)(1-|\zeta|)]\}\exp[s_p\,\tau]. \tag{28}$$

The form of F is identical to that found by Loper & Benton (1970, equations (54) and (55)). Therefore, the fluid visually appears to spin-up in the same manner whether the plates are conductors or insulators, provided the magnetic diffusion regions are thick during the spin-up phase. However, the spin-up time, which is just the negative reciprocal of s_p given by (26), is significantly shorter in the case of conducting plates whenever $\alpha^2\phi > 1+\alpha$.

The strength of the coupling afforded by the conductance of the boundaries is measured by the azimuthal magnetic field perturbation at the fluid–plate interface at $\zeta = 1$, $\xi = d/L$. The dimensionless electric current crossing the interface is twice the azimuthal magnetic field at that point (Benton & Loper 1969, equation (10)). In virtue of (23), it follows that

$$J_z = 2\,\mathrm{Im}[N(d/L,\tau)] = 2\phi(1+\phi E^{\frac{1}{2}})^{-1}\exp(s_p\,\tau). \tag{29}$$

This coupling current is zero for insulating plates ($\phi = 0$). It grows linearly with plate conductance until it eventually becomes limited by the finite conductance of the fluid, reaching a maximum of $(2E^{-\frac{1}{2}})$ as $\phi\to\infty$.

The remainder of this section is devoted to determination of the range of parameters α and ϕ for which (*a*) spin-up is slow or rapid compared with a period of revolution and (*b*) spin-up is accomplished primarily by 'Ekman suction', by electric currents generated within the Ekman–Hartmann boundary layers or by electric currents generated within the bounding plates.

The rapidity of spin-up is easily determined from (26). If $|s_p| < O(1)$ spin-up is comparatively slow, whereas it is rapid if $|s_p| > O(1)$. The line in the (α, ϕ) plane dividing the two domains satisfies

$$[2\alpha^2\phi E^{\frac{1}{2}}/1+\phi E^{\frac{1}{2}}]+E^{\frac{1}{2}}\lambda = 1.$$

To determine the dominant spin-up mechanism, expressions for the various forces acting to spin the fluid up must be obtained. Apart from viscous terms, the imaginary part of (3) is

$$V_\tau = -2U+2\alpha^2E^{\frac{1}{2}}B_\zeta. \quad (30)$$

Thus the time rate of change of the azimuthal velocity equals the sum of the non-magnetic body force $(-2U)$ and the magnetic body force $(2\alpha^2E^{\frac{1}{2}}B_\zeta)$. The ratio of these forces may be calculated using (2), (11), (22)–(25); it is found to be

$$\frac{2\alpha^2E^{\frac{1}{2}}B_\zeta}{-2U} = \frac{\alpha^2\lambda^2(1-\phi^{\frac{1}{2}}E)+2\phi\alpha^2\lambda^2(1+\phi E^{\frac{1}{2}})-2\alpha^4\phi\lambda}{(1+\phi E^{\frac{1}{2}})^2}. \quad (31)$$

If this ratio is less than unity, spin-up is accomplished primarily by Ekman suction. If it is greater than unity, electromagnetic forces are dominant, and the origin of the electric currents causing spin-up (i.e. within the Ekman–Hartmann boundary layer or the boundaries) may be found by inspection of the numerator of (31). Obviously if $\alpha = 0$, the ratio is zero and only non-magnetic forces act. The line dividing the two domains in the (α, ϕ) plane is given by setting the ratio equal to unity.

The five possible domains in the (α, ϕ) plane are shown in figure 2. The parameter α is a measure of the strength of the imposed magnetic field and ϕ is a measure of the boundary conductance. Within domain (i), spin-up is slow compared with a period of revolution and is accomplished primarily by the action of Ekman suction as described by Greenspan & Howard (1963). Note that if $\alpha < O(E^{\frac{1}{4}})$, magnetic effects are never important, regardless of the magnitude of ϕ. Within domain (ii) spin-up is more rapid than in (i) but still is slow compared with a period of revolution. Electric currents generated within the Ekman–Hartmann boundary layer now induce spin-up as described by Loper & Benton (1970). Within domain (iii), spin-up is slow, though more rapid than in (i), and is induced primarily by currents generated within the boundaries. Within domains (iv) and (v), spin-up is rapid compared with a period of revolution and is induced by Ekman–Hartmann boundary-layer currents (domain (iv)) or by boundary currents (domain (v)).

Equation (28) for the fluid velocity function F was obtained by neglecting the branch cuts of (22) which represent boundary-layer formation. Benton & Loper (1970) found that the formation time for the Ekman–Hartmann boundary layer is $2(1+\alpha^4)^{-\frac{1}{2}}$ (see their equation (70)). If the spin-up time is greater than the boundary-layer formation time, (28) should be complete to dominant order. This is the case within domains (i), (ii), (iii) and (iv). However, within domain (v), both spin-up time and boundary-layer formation time are of the order α^{-2}. For

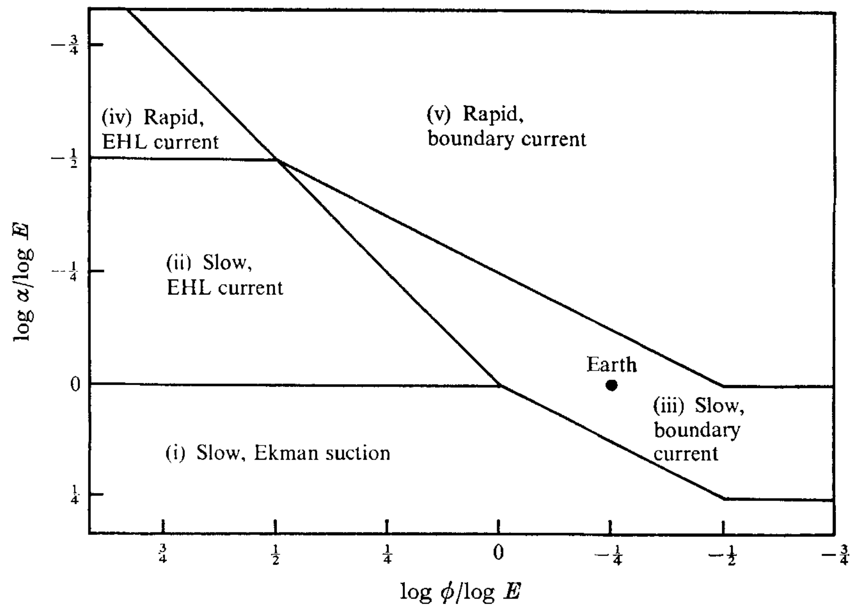

FIGURE 2. Spin-up domains for thick magnetic diffusion layers α, ϕ and E measure magnetic field strength, boundary conductance and fluid viscosity, respectively. (EHL is Ekman–Hartman boundary layer.)

the range of α and ϕ in domain (v) Coriolis forces are unimportant and (22) reduces to

$$\bar{F}(\zeta, s) = \frac{i}{s} - \frac{i\{1-\exp[-E^{-\frac{1}{2}}(2\alpha^2+s)^{\frac{1}{2}}(1-|\zeta|)]\}}{s+2\alpha^2\phi E^{\frac{1}{2}}(1+\phi E^{\frac{1}{2}})^{-1}}.$$

With the aid of Campbell & Foster (1948, §805.3), the inversion is found to be

$$\begin{aligned} F(\zeta, \tau) = \; & i - i\exp(s_p\,\tau)\left\{1-\tfrac{1}{2}\exp\left[E^{-\frac{1}{2}}\left(\frac{2\alpha^2}{1+\phi E^{\frac{1}{2}}}\right)^{\frac{1}{2}}(1-|\zeta|)\right]\right. \\ & \times \operatorname{erf_c}\left[\frac{1-|\zeta|}{2(\tau E)^{\frac{1}{2}}}+\left(\frac{2\alpha^2\tau}{1+\phi E^{\frac{1}{2}}}\right)^{\frac{1}{2}}\right]-\tfrac{1}{2}\exp\left[-E^{-\frac{1}{2}}\left(\frac{2\alpha^2}{1+\phi E^{\frac{1}{2}}}\right)^{\frac{1}{2}}(1-|\zeta|)\right] \\ & \left.\times \operatorname{erf_c}\left[\frac{1-|\zeta|}{2(\tau E)^{\frac{1}{2}}}-\left(\frac{2\alpha^2\tau}{1+\phi E^{\frac{1}{2}}}\right)^{\frac{1}{2}}\right]\right\}. \end{aligned}$$

The boundary-layer functions are now more complicated but the characteristic decay time is still $(-1/s_p)$; the spin-up is still rapid. Also the dominant mechanism of spin-up is still boundary currents.

4. Magnetic diffusion regions thin during spin-up

Assume now that the magnetic Prandtl number δ is sufficiently large that the magnetic diffusion regions diffuse only a short distance compared with the plate separation distance during the spin-up phase. Roughly speaking, this is the case if

$$\delta \gg E\tau_s. \tag{32}$$

To restrict attention to the range of δ of greatest physical interest as well as to simplify the subsequent mathematics, we assume

$$\delta \leqslant O(1). \tag{33}$$

Since the present case of thin magnetic diffusion regions is much more complicated than that of thick considered in the previous section, it is expedient to confine attention to the range of parameter space (to be determined *a posteriori*) for which spin-up is slow; accordingly we assume

$$\tau_s \gg 1 \quad \text{and} \quad |s| \ll 1. \tag{34}$$

The two conditions (32) and (34) give $\delta \gg E$. Assumptions (32)–(34) allow the denominator function D, defined by (14), to be expressed as

$$\begin{aligned} D = {} & 16(1+\alpha^4)^{\frac{1}{2}}E^{-\frac{1}{2}}s[1+(\beta+1/\beta)\phi\delta^{\frac{1}{2}}s^{\frac{1}{2}}+(1+\alpha^4)^{\frac{1}{2}}\phi^2\delta s] \\ & +16\alpha^2\phi(1+\alpha^4)^{\frac{1}{2}}[2+(\beta+1/\beta)\phi\delta^{\frac{1}{2}}s^{\frac{1}{2}}] \\ & +16\left[\beta(1+\alpha^4)^{\frac{1}{2}}+\left(\frac{1-\alpha^2}{(1+\alpha^4)^{\frac{1}{2}}}+1+\alpha^4\right)\phi\delta^{\frac{1}{2}}s^{\frac{1}{2}}+\left(\frac{1}{\beta}-\alpha^2\beta\right)\frac{\phi^2\delta s}{(1+\alpha^4)^{\frac{1}{2}}}\right], \end{aligned} \tag{35}$$

where

$$\beta = [\alpha^2+(1+\alpha^4)^{\frac{1}{2}}]^{\frac{1}{2}}. \tag{36}$$

Equation (35) may be checked in several limits:

(*a*) If $\alpha = 0$, $D = 0$ at $s = -E^{\frac{1}{2}}$ in agreement with Greenspan & Howard (1963).

(*b*) If $\phi = 0$, $D = 0$ at $s = -\beta E^{\frac{1}{2}}$ in agreement with Loper & Benton (1970).

(*c*) If $\delta = 0$, $D = 0$ at $s = -E^{\frac{1}{2}}(\beta+2\alpha^2\phi)$ in agreement with Loper (1970*a*).

The denominator function given by (35) is a polynomial in $s^{\frac{1}{2}}$ with real coefficients. Therefore it has zeros (which yield the poles of $\bar{F}$, $\bar{M}$, $\bar{N}$ and $\bar{P}$) either on the real axis or in conjugate pairs. In each of the limit checks above, the spin-up pole is located on the negative real axis, indicating a negative exponential decay to the final steady state. However, for certain ranges of the parameters (e.g. $\alpha = O(1)$, $|\phi^2\delta s| = O(1)$) the terms in (35) involving $s^{\frac{1}{2}}$ are important. This means that zeros of D occur in conjugate pairs, not on the real axis, and branch cuts contribute to the inversion. This suggests that the spin-up will have an oscillatory behaviour and the fluid will, at some time during the spin-up process, be rotating significantly *faster* than the plates! (In contrast, Greenspan & Howard (1963) found inertial oscillations of very small magnitude in a rotating non-conducting fluid.) Elucidation of this curious phenomenon is one of the purposes of this section.

Despite the simplifications afforded by (32)–(34), the inversion of (10)–(13) is still prohibitively complicated owing to the form of the simplified denominator function (35). Therefore, attention will be restricted further to two special cases for which the analysis is relatively simple:

Case I, $$(1+\alpha)\phi\delta^{\frac{1}{2}}|s^{\frac{1}{2}}| \ll 1; \tag{37}$$

Case II, $$(1+\alpha)\phi\delta^{\frac{1}{2}}|s^{\frac{1}{2}}| \gg 1. \tag{38}$$

The first case demonstrates the influence of plate conductance in decreasing the spin-up time without exciting any oscillations. This repeats in essence the analyses of Loper (1970*a*) and the previous section. The second case illuminates the effect of large plate conductance and thin magnetic diffusion regions. This case yields oscillatory spin-up.†

Case I $$(1+\alpha)\phi\delta^{\frac{1}{2}}|s^{\frac{1}{2}}| \ll 1. \tag{37}$$

In this case $D = 0$, where

$$s_p = -E^{\frac{1}{2}}(\beta+2\alpha^2\phi)+O[(1+\alpha)\alpha^3\phi^{\frac{3}{2}}\delta^{\frac{1}{2}}E^{\frac{3}{4}}] \tag{39}$$

and, to dominant order, the inversion of $\overline{F}$, from (10), is

$$F(\zeta,\tau) = i-i\exp(s_p\tau)\{1-\exp[-E^{-\frac{1}{4}}(\beta+i/\beta)(1-|\zeta|)]\}. \tag{40}$$

This solution is in agreement with (28) for $\phi E^{\frac{1}{2}} \ll 1$ and with (27) of Loper (1970*a*).

The velocity within the magnetic diffusion segions is small, of order

$$\alpha^2\phi^{\frac{3}{2}}\delta^{\frac{1}{2}}E^{\frac{1}{4}} \ll 1.$$

Assumptions (32) and (37) are satisfied if

$$\phi^2\delta \leqslant O(1) \quad \text{and} \quad \phi\alpha^2E^{\frac{1}{2}} \ll 1.$$

In this case, the fluid spins up in a manner described in §2 of Loper & Benton (1970) for $E\tau_s \ll \delta$, the only difference being an additional electric current drawn into the conducting boundaries which increases the coupling between fluid and boundary.

Case II $$(1+\alpha)\phi\delta^{\frac{1}{2}}|s^{\frac{1}{2}}| \gg 1. \tag{38}$$

In effect, this limit is equivalent to the limit $\phi \to \infty$; the expressions below are independent of ϕ. In this limit, the dominant order expression for $\overline{F}(\zeta,s)$ simplifies to

$$\overline{F}(\zeta,s) = \frac{i}{s}-\frac{is^{\frac{1}{2}}}{s^{\frac{3}{2}}+b^{\frac{3}{2}}}\left\{1-\frac{i}{1+\alpha^2}\exp[-E^{\frac{1}{4}}(\beta+i/\beta)(1-|\zeta|)] -\frac{\alpha^2}{i+\alpha^2}\exp\left[-\frac{(2i\delta s)^{\frac{1}{2}}\beta}{E^{\frac{1}{4}}(1+\beta^2)}(1-|\zeta|)\right]\right\}, \tag{41}$$

where $$b^3 = E\delta^{-1}(\beta+1/\beta)^2(1+\alpha^{-4})^{-1}. \tag{42}$$

† The inversion may be accomplished for the case $\alpha \ll 1$, also. This roughly represents the transition between cases I and II. This case is not presented because it is relatively complicated, lengthy and unilluminating.

In order that spin-up be slow, b must be small compared with unity. Also, it is necessary to suppose that α is sufficiently large that the present spin-up mechanism is more effective than the non-magnetic spin-up mechanism of convergence of fluid into the Ekman boundary layers. This requirement limits α to the range

$$(1 \gg)\delta^{\frac{1}{2}}E^{\frac{1}{4}} \ll \alpha^{\frac{1}{2}} \ll \delta E^{-1}(\gg 1). \tag{43}$$

Condition (43) is not overly restrictive inasmuch as it admits values of α both much greater than and much less than unity.

It is noteworthy that (41) admits significant velocities within the magnetic diffusion regions during spin-up. In fact, if $\alpha \gg 1$, the magnetic diffusion regions satisfy the viscous boundary conditions and the Ekman–Hartmann boundary layers are no longer necessary. In this extreme $(\phi, \alpha \to \infty)$ currents flowing radially inward within the magnetic diffusion regions interact with the axial applied magnetic field to produce a strong hydromagnetic body force in the azimuthal direction. This body force causes the fluid within the magnetic diffusion regions to rotate with the speed of the plates, obviating the need for Ekman–Hartmann boundary layers.

Equation (41) may be inverted but the result is too cumbersome to be of much use. Consider instead the radial and azimuthal velocities outside both layers. From (10), the dominant real and imaginary components of $\overline{F}$ are

$$\overline{F}_0(s) = \frac{-b^{\frac{3}{2}}}{2(s^{\frac{3}{2}}+b^{\frac{3}{2}})} + i\left[\frac{1}{s} - \frac{s^{\frac{1}{2}}}{s^{\frac{3}{2}}+b^{\frac{3}{2}}}\right]. \tag{44}$$

This may be directly inverted using residues and contour integration to yield

$$U_0(\tau) = -\tfrac{1}{3}b\exp\left(-\tfrac{1}{2}b\tau\right)\cos\left(\tfrac{1}{2}b\tau\surd 3\right) - (b/\surd 3)\exp\left(-\tfrac{1}{2}b\tau\right)\sin\left(\tfrac{1}{2}b\tau\surd 3\right) + (b/\pi)\int_0^\infty \xi^4(1+\xi^6)^{-1}\exp\left(-b\tau\xi^2\right)d\xi \tag{45}$$

and

$$V_0(\tau) = 1 - \tfrac{4}{3}\exp\left(-\tfrac{1}{2}b\tau\right)\cos\left(\tfrac{1}{2}b\tau\surd 3\right) + (2/\pi)\int_0^\infty \xi^2(1+\xi^6)^{-1}\exp\left(-\xi^2 b\tau\right)d\xi. \tag{46}$$

The radial velocity U_0 and the azimuthal velocity V_0 are plotted versus time in figure 3. The flow is geostropic, i.e.

$$dV_0/d\tau + 2U_0 = 0$$

in the inviscid current-free core. At the instant the fluid has spun up ($b\tau = 1{\cdot}65$ in figure 3) the radial velocity is still 70 % of its maximum value. Consequently, the fluid overshoots its final spin-up value by approximately 30 % for $b\tau \approx 3$. In other words, the fluid actually spins faster than the plates during part of the spin-up phase. Also, it in interesting that the azimuthal velocity approaches its final steady value from above; that is, $V_0(\tau) > 1$ for $1/b \ll \tau < \infty$.

The system responds to the initial perturbation in a manner characteristic of under-damped mechanical systems. In the present case, the inertia of the system appears to be provided by the flow in meridional planes, the restoring force by the pressure gradient and the dissipation by resistivity of the magnetic diffusion regions.

In the non-magnetic spin-up problem considered by Greenspan & Howard (1963), the meridional circulation is coupled to the viscous Ekman boundary layers and fluid viscosity strongly inhibits inertial oscillations. Similarly, in the insulating boundary problem of Loper & Benton (1970), the circulation is coupled to two dissipative mechanisms, viscous and resistive, which prevent significant oscillations from occurring. In the present problem, if α and ϕ are

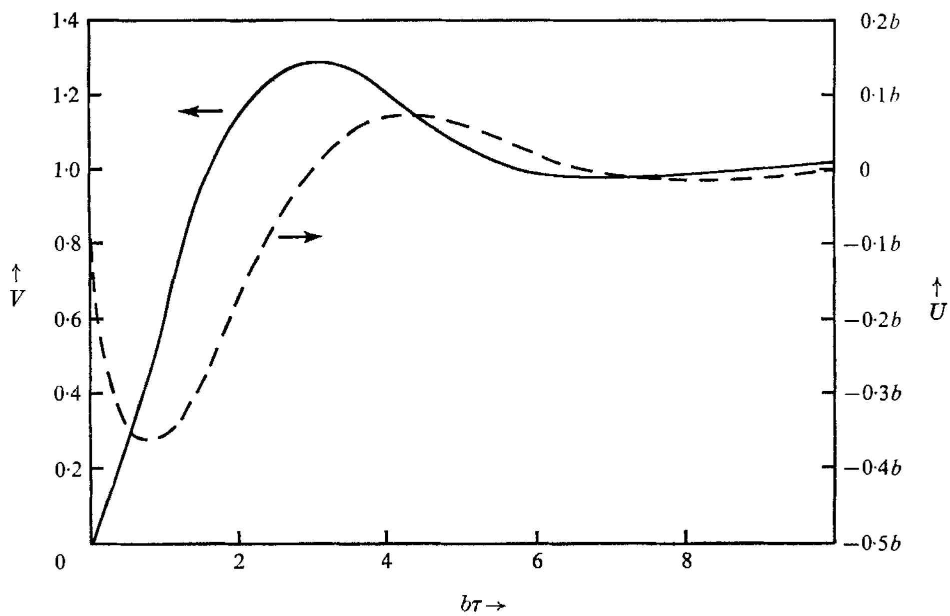

FIGURE 3. Radial (dashed line) and azimuthal (solid line) perturbation velocities for thin magnetic diffusion regions and large boundary conductance.

sufficiently large, viscosity of the fluid plays a negligible role in the spin-up process. Also if ϕ is very large, electrical resistance within the plates is negligible. If the magnetic diffusion regions are thick during spin-up as in § 3, the resistivity of the fluid is sufficiently strong to produce only negligibly small oscillations. On the other hand, if the magnetic diffusion regions are thin during spin-up, resistivity of the fluid is low enough that the system becomes underdamped and significant oscillations of the fluid may occur. This mode of spin-up cannot be predicted from the corresponding single plate analysis of Loper (1970*a*).

Oscillatory spin-up of a contained rotating fluid is not uniquely a hydromagnetic phenomenon. The spin-up of a fluid bounded axially by anisotropic porous media also exhibits oscillations if the resistance to flow in the azimuthal direction is much greater than that in meridional planes. In this case, the anistropic porous media play the role of the magnetic diffusion regions but the inertia is still provided by the meridional flow. The conceptually simpler, non-magnetic oscillatory spin-up is currently under investigation, both analytically and experimentally, by G. Buzyna and the author.

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REFERENCES

BENTON, E. R. & LOPER, D. E. 1969 On the spin-up of an electrically conducting fluid. Part 1. The unsteady hydromagnetic Ekman–Hartmann boundary-layer problem. *J. Fluid Mech.* **39**, 561–586.

CAMPBELL, G. A. & FOSTER, R. N. 1948 *Fourier Integrals for Practical Applications*. Van Nostrand.

COWLING, T. G. 1957 *Magnetohydrodynamics*. Wiley.

GREENSPAN, H. P. & HOWARD, L. N. 1963 On a time-dependent motion of a rotating fluid. *J. Fluid Mech.* **17**, 385–404.

LOPER, D. E. 1970*a* On the steady hydromagnetic boundary layer near a rotating, electrically conducting plate. *Phys. Fluids*, **13**, 2999–3002.

LOPER, D. E. 1970*b* On the unsteady hydromagnetic perturbations at the earth's core-mantle interface. *Phys. Earth & Planetary Interiors*, **4**, 129–137.

LOPER, D. E. & BENTON, E. R. 1970 On the spin-up of an electrically conducting fluid. Part 2. Hydromagnetic spin-up between infinite flat, insulating plates. *J. Fluid Mech.* **43**, 785–800